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Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects

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Résumé

Considering the three-dimensional Navier–Stokes equations with a free moving surface boundary condition and hydrostatic approximation, we study the derivation, with asymptotic analysis, of a new two-dimensional viscous shallow water model in rotating framework, with irregular topography, linear and quadratic bottom friction terms and capillary effects. A new formulation of the viscous effects, consistent with a previous one-dimensional analysis, is obtained. Finally, we propose some simple numerical experiments in order to validate the proposed model.

Keywords: Asymptotic analysis; Shallow water equations; Viscosity; Friction; Capillary effects

1. Introduction

These notes are devoted to the derivation of a new two-dimensional shallow water model, in the spirit of the method proposed for the one-dimensional case by Gerbeau and Perthame in [1]. Non-linear shallow water equations (NLSW) model the dynamics of a shallow, rotating layer of homogeneous incompressible and inviscid fluid and are typically used to describe vertically averaged flows in three dimensional domains, in terms of horizontal velocity and depth variation. This set of equations is particularly well-suited for the study and numerical simulations of a large class of geophysical phenomena, such as rivers, coastal domains, oceans, or even run-off or avalanches when modified with adapted source terms. These models are also extensively used in the field of hydraulic applications. The usual conservative form of NLSW equations introduced in [2], without viscosity and capillary effects, is written as a first order hyperbolic system with various source terms as bed slope or bottom friction terms. The derivation of this first order system is classical [3,4]. In the fully two-dimensional system, the Coriolis effects are also included. However, for particular applications, like in oceanography for the numerical computation of long period large eddies evolutions, additional viscosity or special friction terms are needed.

The new model introduced here is obtained from an asymptotic analysis of the non-dimensional and incompressible Navier–Stokes–Coriolis equations with a large-scale assumption and hydrostatic approximation in a rotating sub-domain of $\mathbb{R}^3$. The method for the one-dimensional case proposed in [1] and recently extended for the derivation of

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a multilayer one-dimensional shallow water model in [5] is developed in the two-dimensional case. The free moving surface boundary condition for the Navier–Stokes equations is completed with normal stress continuity and capillary effects at the air-fluid interface. The bottom is considered as irregular with an assumption of slow variations. A study of gravity driven shallow water flows without any restriction on the topography can be found for instance in [6] and [7]. Furthermore, we introduce here different kinds of friction terms in the boundary conditions, namely linear and quadratic wall-laws.

As emphasized in [8] and [9], the introduction of the third-order surface-tension term induced by capillary effects and of the quadratic bottom friction term is of a great utility in the proof of existence of weak-solutions for this particular model. Furthermore, the quadratic formulation of the bottom friction allows us to extend the validity of this set of NLSW equations to coastal hydrodynamic simulations by the use of Manning–Chezy or Strickler numerical formulation (see [10]). The two-dimensional framework also highlights the possible existence of a particular viscous term, which was only identified as a modified coefficient in the one-dimensional case.

The derivation is realized in two steps, following [1]. First we derive a shallow water model (5.9) with Coriolis effects and irregular bottom with a small laminar friction term, resulting from a first order analysis, thanks to the large-scale assumption and the hydrostatic approximation. In a second part, we perform a parabolic correction of the horizontal velocity in order to take into account the vertical variability of the phenomenon. A more accurate asymptotic analysis provides a new model (5.19) with a particular viscous term, capillary effects and linear and quadratic drag terms with water depth dependent coefficients. In the last section, some academical numerical experiments are performed in order to compare the effects of this new viscous formulation with the formulation expected from the one-dimensional case.

2. The Navier–Stokes–Coriolis system and boundary conditions

We briefly introduce the classical set of Navier–Stokes–Coriolis equations which is considered as the basis of the following analysis. Well-suited boundary conditions are proposed for the free surface and the bottom boundaries.

2.1. The system

We start from the rotating Navier–Stokes equations for incompressible homogeneous fluids, namely the Navier–Stokes–Coriolis system, evolving in a sub-domain of $\mathbb{R}^3$. These equations are studied in a frame which rotates with the planetary angular velocity Ω, considered as constant. For homogeneous and incompressible fluids motions, the condition of mass conservation reduces to the incompressibility condition $\operatorname{div}\mathbf{U} = 0$. Expressing the motions in terms of quantities which are observed in the rotating frame, we obtain the following system, in a local frame (x, y, z) where (x, y) defines a tangential surface to the fluid domain (see Fig. 1):

$$\begin{cases} \operatorname{div}\mathbf{U} = 0, \\ \partial_t\mathbf{U} + \operatorname{div}(\mathbf{U}\otimes\mathbf{U}) = \operatorname{div}\sigma(\mathbf{U}) + F_c(\mathbf{U}) + \mathbf{g}, \end{cases} \tag{2.1}$$

where

- $\mathbf{U} = (u, v, w)$ is the velocity seen in the rotating frame.
- $\operatorname{div}\sigma(\mathbf{U})$ are the viscosity forces and σ the total stress tensor. An explicit form for this tensor is:

  $$\sigma(\mathbf{U}) = -p\mathcal{I} + 2\mu D(\mathbf{U}), \tag{2.2}$$

  where $p(t, x, y, z)$ represents the local pressure in the fluid, μ is the dynamical viscosity and $D(\mathbf{U})$ the viscosity tensor:

  $$D_{ij}(\mathbf{U}) = \frac{1}{2}(\partial_j u_i + \partial_i u_j) \quad \text{for } 1 \leqslant i, j \leqslant 3, \tag{2.3}$$

  where $(u_1, u_2, u_3) = (u, v, w)$ and $\partial_1 = \partial_x$, $\partial_2 = \partial_y$, $\partial_3 = \partial_z$. Note that the eddy viscosity is neglected here, since no turbulence model is added for the stress tensor. If the turbulent contribution to the momentum equations has received attention in hydraulics, the situation is different is coastal engineering or in the study of flood propagation, where turbulence modeling is often not considered as an important matter. In some anisotropic approaches,

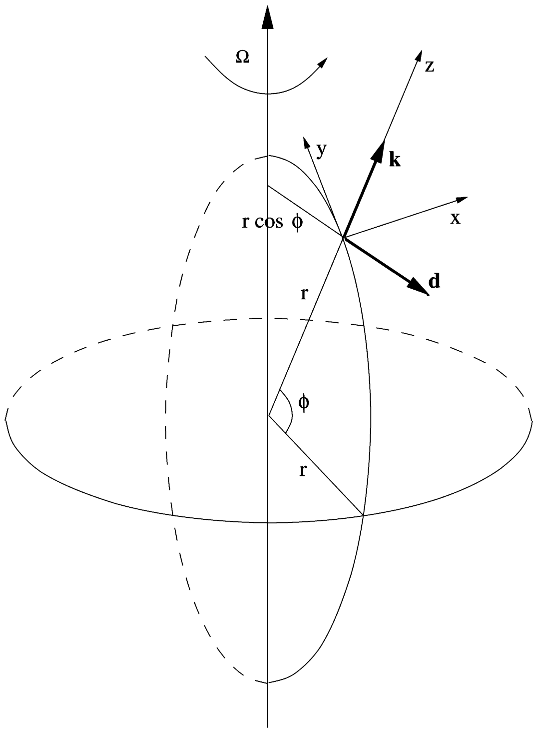

Fig. 1. Spherical coordinates and local frame.

constant eddy viscosity coefficients are introduced (refer to [11,12]) which seem to be more tuning parameters than a physical description of the turbulence characteristics.

- $F_c(\mathbf{U}) = (fv, -fu, 0)$ and $\mathbf{g} = (0, 0, -g)$ stands respectively for the Coriolis effect, with the Coriolis coefficient f, and the effective gravitational acceleration, oriented along the radial direction, from the position of the fluid element to the center of the earth.

Remark 1. We recall that the Coriolis coefficient f can be regarded as the local component of the planetary vorticity normal to the earth surface. A classical approximation $f = 2\Omega \sin \Phi$, where Φ is the latitude considered as constant, can be obtained by focusing our attention on large-scale motions only and considering the nearly horizontal character of the fluid trajectories [10].

Remark 2. The effective gravitational acceleration (oriented downward) is usually the combination of the centrifugal acceleration and the earth's Newtonian gravitational attraction [10].

In the sequel, we will focus our attention on this system for

$$t > 0, (x, y) \in \mathbb{R} \quad \text{and} \quad d(x, y) \leqslant z \leqslant \xi(t, x, y),$$

where $z = \xi(t, x, y)$ represents the local water elevation from the surface $z = 0$ to the air/fluid interface and $z = d(x, y)$ represents the description of the topography variations. We also introduce at once $h(t, x, y)$ which describes the total length of the water column located at the (x, y) coordinate (see Fig. 2) with:

$$h(t, x, y) = \xi(t, x, y) - d(x, y). \tag{2.4}$$

2.2. Boundary conditions

Well-suited boundary conditions are respectively introduced for the bottom and the air–fluid interface boundaries.

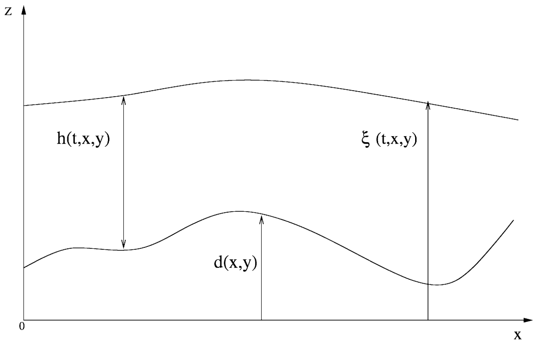

Fig. 2. Mean variables.

2.2.1. *Normal stress continuity and surface tension*

On the free surface, we assume a normal-stress continuity condition with surface tension at the air/fluid interface, considering the air viscosity as negligible:

$$\sigma(\mathbf{U})\cdot n_s - \beta\kappa(t,x,y)n_s = -p_o n_s \quad \text{at } z=\xi(t,x,y), \tag{2.5}$$

where n_s is the outward normal to the free surface, defined with $n_s = \nabla S/|\nabla S|$ and $S = z - \xi(t,x,y)$, β is a capillary coefficient, κ the mean curvature of the surface at point (x,y) and $p_o(x,y)$ the atmospheric pressure at the surface. We obtain the following expression:

$$n_s = \frac{1}{\sqrt{1+|\nabla\xi|^2}}\begin{pmatrix}-\partial_x\xi\\ -\partial_y\xi\\ 1\end{pmatrix}.$$

2.2.2. *Boundary conditions at the bottom*

At the bottom, we assume a wall-law with linear and quadratic terms (see [10,9]), which respectively refer to laminar and turbulent friction phenomena:

$$\big(\sigma(\mathbf{U})\cdot n_b\big)\cdot\tau_{b_i} = k_l(\mathbf{U}\cdot\tau_{b_i}) + k_t\big(h|\mathbf{U}|\mathbf{U}\cdot\tau_{b_i}\big) \quad \text{at } z=d(x,y), \tag{2.6}$$

where k_l and k_t are the laminar and the turbulent friction coefficients. The particular forms of these parameterizations of small scales are suggested by empirical laws, which were originally obtained for steady state flow conditions, like the usual quadratic formulae of the Manning or Chezy type (refer to [13]). Note that Slinn et al. [14] have used a linear friction term in simulations of alongshore currents near beaches. It is worth mentioning that the friction coefficients are almost universally considered constant, disregarding variations in bed roughness (refer to Aronica et al. [15] for variable friction coefficient applications). More complex laws can also be formulated to modelize analogous problems in the case of granular media [6].

The outward normal is defined with:

$$n_b = \frac{1}{\sqrt{1+|\nabla d|^2}}\begin{pmatrix}-\partial_x d\\ -\partial_y d\\ 1\end{pmatrix},$$

and $(\tau_{b_i})_{i=1,2}$ is a basis of the tangential surface:

$$\tau_{b_1} = \frac{1}{|\nabla d|}\begin{pmatrix}-\partial_y d\\ \partial_x d\\ 0\end{pmatrix} \quad \text{and} \quad \tau_{b_2} = \frac{1}{\sqrt{|\nabla d|^2+|\nabla d|^4}}\begin{pmatrix}-\partial_x d\\ -\partial_y d\\ -|\nabla d|^2\end{pmatrix}. \tag{2.7}$$

We complete this condition with a no-penetration condition at the bottom:

$$\mathbf{U}\cdot n_b = 0 \quad \text{at } z=d(x,y). \tag{2.8}$$

2.3. Indicator function

Following the idea developed in [1], let ϕ denotes an indicator function which allows us to define precisely the fluid region at time t:

$$\phi(t,x,y,z)=\begin{cases}1 & \text{for } d(x,y)\leqslant z\leqslant \xi(t,x,y),\\ 0 & \text{otherwise.}\end{cases} \tag{2.9}$$

The mass conservation equation becomes, using the incompressibility relation:

$$\partial_t\phi+\partial_x(\phi u)+\partial_y(\phi v)+\partial_z(\phi w)=0. \tag{2.10}$$

3. The nondimensionalized system

In this section, we briefly consider some characteristic scales which applied for the shallow water model. Then, a nondimensionalized set of equations is obtained and the usual hydrostatic approximation is assumed.

3.1. Some considerations about vertical scales in oceanic movements

As it has been previously exposed, the NLSW equations with viscosity, friction terms and topography are extensively used in the fields of oceanography and particularly for the numerical simulation of oceanic movements. Although the depth of the fluid varies in space and time, we assume that characteristic scale for the depth can be chosen. This average thickness H of the oceans is nearly 4 km whereas their horizontal characteristic value is about 4000 km. So the ratio of the vertical scale to the horizontal one, the so-called aspect ratio, is about $1/1000$. The situation is analogous in coastal domain where the characteristic scale for vertical variations is reduced to 100 m and the horizontal one is about 100 km. In this context, we shall assume that the vertical movements and variations are very small compared to the horizontal ones. The continuity equation allows us in the same way to estimate the ratio of the vertical and horizontal velocity scales, respectively W and V:

$$\frac{W}{U}\approx\frac{H}{L}\approx\frac{1}{1000}.$$

3.2. Dimensionless quantities

According to the previous section, we use in the sequel the thin-layer assumption and introduce a "small" parameter $\varepsilon=H/L=W/V$ where H, L, W, V are respectively the characteristic scales for the vertical and the horizontal dimension of the fluid domain of interest and for the vertical and the horizontal velocity. We then introduce some characteristic dimensions: $T=L/V$ and $P=V^2$ for the time and the pressure.

The dimensionless quantities are temporarily noted with a $\tilde{}$ and are defined as follows: $\tilde u=u/V$, $\tilde v=v/V$, $\widetilde w=w/W$, $\tilde x=x/L$, $\tilde y=y/L$, $\tilde z=z/H$, $\tilde t=t/T$ and $\tilde p=p/P$. Some dimensionless numbers are also introduced: the Reynolds number $R_e=VL/\mu$, the Froude number $F_r=V/\sqrt{gH}$ and the Rossby number $R_o=V/fL$. Finally, let us define the modified laminar and quadratic friction coefficient $k_l'=k_l/V$ and $k_t'=k_tL$.

In what follows, we mostly focus on the horizontal velocity. We thus distinguish the horizontal components (u,v) and the vertical one w. The tensor $\sigma(U)$ defined in (2.2) will be denoted by σ for the sake of clarity and we introduce $U=(u,v)$. It is also convenient to introduce the tensor $D_h(U)$ with:

$$\bigl(D_h(U)\bigr)_{ij}=\frac{1}{2}(\partial_j u_i+\partial_i u_j)\quad \text{for } 1\leqslant i,j\leqslant 2, \tag{3.1}$$

where $u_1=u$, $u_2=v$, $\partial_1=\partial_x$ and $\partial_2=\partial_y$. Finally we denote by $U^\perp$ the following quantity: $U^\perp=(-v,u)$. The differential operators ∇_{xy}, div_{xy} and Δ_{xy} refer to horizontal derivations.

Dropping the $\tilde{}$, the nondimensionalized Navier–Stokes system becomes:

$$\begin{cases}\mathrm{div}_{xy}\,U+\partial_z w=0,\\ \partial_t U+\mathrm{div}\,x(U\otimes U)+\partial_z(wU)+\nabla_{xy}p+\dfrac{U^\perp}{R_o}=\dfrac{1}{R_e}\left(2\,\mathrm{div}_{xy}\bigl(D_h(U)\bigr)+\dfrac{1}{\varepsilon^2}\partial_z^2U+\nabla_{xy}(\partial_z w)\right),\\ \varepsilon^2\bigl(\partial_t w+\mathrm{div}_{xy}(wU)+\partial_z\bigl(w^2\bigr)\bigr)+\partial_z p=-\dfrac{1}{F_r^2}+\dfrac{1}{R_e}\bigl(\partial_z(2\,\mathrm{div}_{xy}\,U)+\varepsilon^2\Delta_{xy}w+2\partial_z^2 w\bigr).\end{cases} \tag{3.2}$$

Boundary conditions are also written in a dimensionless form:

$$\begin{cases} (p-p_o+\beta\kappa)\nabla_{xy}\xi + \dfrac{1}{R_e}\left(\dfrac{1}{\varepsilon^2}\partial_z U + \nabla_{xy} w - 2D_h(U)\cdot\nabla_{xy}\xi\right) = 0 \quad \text{at } z=\xi(t,x,y), \\ p-p_o+\beta\kappa + \dfrac{1}{R_e}\big(\nabla_{xy}\xi\cdot\big(\partial_z U + \varepsilon^2\nabla_{xy} w\big) - 2\partial_z w\big) = 0 \quad \text{at } z=\xi(t,x,y), \end{cases} \tag{3.3}$$

and:

$$\begin{cases} 1/\big(\varepsilon^2 V^2\big)(\sigma\cdot n_b)\cdot\tau_{b_1} = -\dfrac{1}{\varepsilon}\big(k_l' + \varepsilon k_t' h|U|\big)\big(U^\perp\cdot\nabla_{xy} d\big) \quad \text{at } z=d(x,y), \\ 1/\big(\varepsilon^2 V^2\big)(\sigma\cdot n_b)\cdot\tau_{b_2} = -\dfrac{1}{\varepsilon}\big(k_l' + \varepsilon k_t' h|U|\big)\big(U\cdot\nabla_{xy} d + \varepsilon^2 w|\nabla_{xy} d|^2\big) \quad \text{at } z=d(x,y), \\ U\cdot\nabla_{xy} d - w = 0 \quad \text{at } z=d(x,y). \end{cases} \tag{3.4}$$

The hydrostatic approximation, which consists in dropping terms of second order of magnitude in ε in the relations (see [1,16] and [10]), enables us to go ahead in the derivation. Actually, the nearly horizontal nature of the fluid evolution induces so small vertical accelerations that the Archimedian principle for a static fluid is applicable. The system (3.2)–(3.4) becomes:

$$\begin{cases} \operatorname{div}_{xy} U + \partial_z w = 0, \\ \partial_t U + \operatorname{div}_{xy}(U\otimes U) + \partial_z(wU) + \nabla_{xy} p + \dfrac{U^\perp}{R_o} = \dfrac{1}{R_e}\left(2\operatorname{div}_{xy}\big(D(U)\big) + \dfrac{1}{\varepsilon^2}\partial_z^2 U + \nabla_{xy}(\partial_z w)\right), \\ \partial_z p = -\dfrac{1}{F_r^2} + \dfrac{1}{R_e}\big(\partial_z(\operatorname{div}_{xy} U) + 2\partial_z^2 w\big), \end{cases} \tag{3.5}$$

at the free surface boundary:

$$\begin{cases} (p-p_o+\beta\kappa)\nabla_{xy}\xi + \dfrac{1}{R_e}\left(\dfrac{1}{\varepsilon^2}\partial_z U + \nabla_{xy} w - 2D(U)\cdot\nabla_{xy}\xi\right) = 0 \quad \text{at } z=\xi(t,x,y), \\ p-p_o+\beta\kappa + \dfrac{1}{R_e}(\nabla_{xy}\xi\cdot\partial_z U - 2\partial_z w) = 0 \quad \text{at } z=\xi(t,x,y), \end{cases} \tag{3.6}$$

and at the bottom:

$$\begin{cases} 1/\big(\varepsilon^2 V^2\big)(\sigma\cdot n_b)\cdot\tau_{b_1} = -\dfrac{1}{\varepsilon}\big(k_l' + \varepsilon k_t' h|U|\big)\big(U^\perp\cdot\nabla_{xy} d\big) \quad \text{at } z=d(x,y), & \text{(i)} \\ 1/\big(\varepsilon^2 V^2\big)(\sigma\cdot n_b)\cdot\tau_{b_2} = \dfrac{1}{\varepsilon}\big(k_l' + \varepsilon k_t' h|U|\big)(U\cdot\nabla_{xy} d) \quad \text{at } z=d(x,y), & \text{(ii)} \\ U\cdot\nabla_{xy} d - w = 0 \quad \text{at } z=d(x,y). & \text{(iii)} \end{cases} \tag{3.7}$$

The solution of this system always depends on ε, owing to the horizontal momentum conservation relation in (3.5). Keeping in mind that we are looking for a solution of order O(1), we emphasize that (3.6) gives:

$$\frac{1}{R_e}\partial_z U_{|z=\xi} = -\varepsilon^2\big((p-p_o+\beta\kappa)\nabla_{xy}\xi - 2D(U)\cdot\nabla_{xy}\xi + \nabla_{xy} w\big)_{|z=\xi}. \tag{3.8}$$

This term cannot be neglected, since we have the $(1/\varepsilon^2)\partial_z^2 U$ term in the horizontal momentum relation.

Moreover, we have, always from relation (3.6):

$$\left(p - p_o - \frac{2}{R_e}\partial_z w + \beta\kappa\right)_{|z=\xi} = -\frac{1}{R_e}(\nabla_{xy}\xi\cdot\partial_z U)_{|z=\xi}. \tag{3.9}$$

We notice that the left hand part of (3.9) appears as a O(1) term when the relations (3.5) are integrated along the vertical direction. Using (3.8) and recalling that we have $(1/R_e)\partial_z U_{|z=\xi} = \mathrm{O}(\varepsilon^2)$, we obtain:

$$\left(p - p_o - \frac{2}{R_e}\partial_z w + \beta\kappa\right)_{|z=\xi} = \mathrm{O}\big(\varepsilon^2\big). \tag{3.10}$$

Integrating the relation (3.5) for the pressure and using (3.10) gives:

$$p(t,x,y,z) - p_o + \beta\kappa(t,x,y) = \frac{1}{F_r^2}(\xi - z) + \frac{1}{R_e}\left(\int_\xi^z \partial_x\big(\partial_z u(t,x,y,\eta)\big)\,\mathrm{d}\eta + \int_\xi^z \partial_y\big(\partial_z v(t,x,y,\eta)\big)\,\mathrm{d}\eta + 2\partial_z w(t,x,y,z)\right) + \mathrm{O}\big(\varepsilon^2\big).$$

Finally, thanks to the incompressibility relation, we get:

$$p(t,x,y,z) - p_o + \beta\kappa(t,x,y) = \frac{1}{F_r^2}(\xi - z) - \frac{1}{R_e}\big(\partial_x u(t,x,y,z) + \partial_x u(t,x,y,\xi) + \partial_y v(t,x,y,z) + \partial_y v(t,x,y,\xi)\big) + \mathrm{O}\big(\varepsilon^2\big). \tag{3.11}$$

4. The vertically averaged nondimensionalized system

4.1. Free surface condition

If (2.10) is integrated between $z = d(x,y)$ and $z = +\infty$, we get by Leibniz formula:

$$\partial_t \int_d^\infty \phi\,\mathrm{d}z + \partial_x\left(\int_d^\infty \phi u\,\mathrm{d}z\right) + \partial_y\left(\int_d^\infty \phi v\,\mathrm{d}z\right) + [\phi w]_d^\infty + (u\partial_x d + v\partial_y d)_{|z=d} = 0.$$

Using the no-penetration condition (2.8) and keeping in mind the definition of the indicator function ϕ (2.9), we obtain:

$$\partial_t h(t,x,y) + \partial_x\left(\int_d^\xi u(t,x,y,z)\,\mathrm{d}z\right) + \partial_y\left(\int_d^\xi v(t,x,y,z)\,\mathrm{d}z\right) = 0, \tag{4.1}$$

where $h(t,x,y)$ is defined by (2.4).

If (2.10) is integrated in a different way, namely between $z = d(x,y)$ and $z = \xi(t,x,y)$, we obtain:

$$\partial_t h + \partial_x\left(\int_d^\xi u\,\mathrm{d}z\right) + \partial_y\left(\int_d^\xi v\,\mathrm{d}z\right) - (\partial_t\xi + u\partial_x\xi + v\partial_y\xi - w)_{|z=\xi} + (u\partial_x d + v\partial_y d - w)_{|z=d} = 0.$$

Finally, gathering these results and using again condition (2.8), we get:

$$(\partial_t\xi + U\cdot\nabla_{xy}\xi - w)_{|z=\xi} = 0. \tag{4.2}$$

4.2. Momentum equation

Integrating the horizontal momentum relation (3.5) for $d(x,y) \leqslant z \leqslant \xi(t,x,y)$ gives:

$$\begin{aligned}
&\partial_t\left(\int_d^\xi U\,\mathrm{d}z\right) + \mathrm{div}_{xy}\left(\int_d^\xi (U\otimes U)\,\mathrm{d}z\right) + \nabla_{xy}\left(\int_d^\xi p\,\mathrm{d}z\right) + \frac{1}{R_o}\int_d^\xi U^\perp\,\mathrm{d}z \\
&\quad - U(\partial_t\xi + U\cdot\nabla_{xy}\xi - w)_{|z=\xi} + U(U\cdot\nabla_{xy}d - w)_{|z=d} \\
&= \mathrm{div}_{xy}\left(\int_d^\xi \frac{2}{R_e}D_h(U)\,\mathrm{d}z\right) + \frac{1}{R_e}\left(\left(\frac{1}{\varepsilon^2}\partial_z U + \nabla_{xy}w\right)_{|z=\xi} - \left(\frac{1}{\varepsilon^2}\partial_z U + \nabla_{xy}w\right)_{|z=d}\right) \\
&\quad + \left(p\nabla_{xy}\xi - \frac{2}{R_e}D_h(U)\cdot\nabla_{xy}\xi\right)_{|z=\xi} - \left(p\nabla_{xy}d - \frac{2}{R_e}D_h(U)\cdot\nabla_{xy}d\right)_{|z=d}.
\end{aligned}$$

Using (4.2) together with the boundary conditions at the bottom (2.8) and at the free surface (3.6) we obtain:

$$\begin{aligned}
&\partial_t\left(\int_d^{\xi} U\,\mathrm{d}z\right)+\mathrm{div}_{xy}\left(\int_d^{\xi}(U\otimes U)\,\mathrm{d}z\right)+\nabla_{xy}\left(\int_d^{\xi} p\,\mathrm{d}z\right)+\frac{1}{R_o}\int_d^{\xi} U^{\perp}\,\mathrm{d}z\\
&=\mathrm{div}_{xy}\left(\int_d^{\xi}\frac{2}{R_e}D_h(U)\,\mathrm{d}z\right)+(p_o-\beta\kappa)\nabla_{xy}\xi\\
&\quad-\left(p\nabla_{xy}d-\frac{2}{R_e}D_h(U)\cdot\nabla_{xy}d+\frac{1}{R_e}\left(\frac{1}{\varepsilon^2}\partial_z U+\nabla_{xy}w\right)\right)_{|z=d}.
\end{aligned}\tag{4.3}$$

It remains to use the friction boundary conditions at the bottom (3.7) to conclude this first step in the derivation. Developing these relations, the following linear combination ($\partial_y d\times(3.7(\mathrm{i}))+\partial_x d\times(3.7(\mathrm{ii}))$ gives:

$$\begin{aligned}
&\left(\big(-p+(D_h)_{xx}\big)\partial_x d+(D_h)_{xy}\partial_y d-\frac{1}{R_e}\left(\frac{1}{\varepsilon^2}\partial_z u+\partial_x w\right)\right)_{|z=d}\\
&=-\frac{k_l'}{\varepsilon}u_{|z=d}-k_t h|U|u_{|z=d}-\partial_x d\left(p-\frac{2}{R_e}\partial_z w\right)_{|z=d}+\frac{1}{R_e}\partial_x d(\nabla_{xy}d\cdot\partial_z U)_{|z=d}.
\end{aligned}$$

In the same way, with the combination ($\partial_y d\times(3.7(\mathrm{ii}))-\partial_x d\times(3.7(\mathrm{i}))$, we obtain the second relation:

$$\begin{aligned}
&\left(\big(-p+(D_h)_{yy}\big)\partial_y d+(D_h)_{xy}\partial_x d-\frac{1}{R_e}\left(\frac{1}{\varepsilon^2}\partial_z v+\partial_y w\right)\right)_{|z=d}\\
&=-\frac{k_l'}{\varepsilon}v_{|z=d}-k_t h|U|v_{|z=d}-\partial_y d\left(p-\frac{2}{R_e}\partial_z w\right)_{|z=d}+\frac{1}{R_e}\partial_y d(\nabla_{xy}d\cdot\partial_z U)_{|z=d}.
\end{aligned}$$

Gathering these two scalar relations, we obtain the vectorial one:

$$\begin{aligned}
&\left(-p\nabla_{xy}d+\frac{2}{R_e}D_h(U)\cdot\nabla_{xy}d-\frac{1}{R_e}\left(\frac{1}{\varepsilon^2}\partial_z U+\nabla_{xy}w\right)\right)_{|z=d}\\
&=-\frac{k_l'}{\varepsilon}U_{|z=d}-k_t h|U|U_{|z=d}-\nabla_{xy}d\left(p-\frac{2}{R_e}\partial_z w\right)_{|z=d}+\frac{1}{R_e}\nabla_{xy}d(\nabla_{xy}d\cdot\partial_z U)_{|z=d}.
\end{aligned}\tag{4.4}$$

This relation (4.4) can be directly used in Eq. (4.3) and we finally obtain:

$$\begin{aligned}
&\partial_t\left(\int_d^{\xi} U\,\mathrm{d}z\right)+\mathrm{div}_{xy}\left(\int_d^{\xi}(U\otimes U)\,\mathrm{d}z\right)+\nabla_{xy}\left(\int_d^{\xi} p\,\mathrm{d}z\right)+\frac{1}{R_o}\int_d^{\xi} U^{\perp}\,\mathrm{d}z\\
&=-\frac{k_l'}{\varepsilon}U_{|_{z=d}}-k_t h|U|U_{|_{z=d}}+\mathrm{div}_{xy}\left(\int_d^{\xi}\frac{2}{R_e}D_h(U)\,\mathrm{d}z\right)+(p_o-\beta\kappa)\nabla_{xy}\xi\\
&\quad-\nabla_{xy}d\left(p-\frac{2}{R_e}\partial_z w\right)_{|z=d}+\frac{1}{R_e}\nabla_{xy}d(\nabla_{xy}d\cdot\partial_z U)_{|z=d}.
\end{aligned}\tag{4.5}$$

5. The shallow water system

In this section, the derivation of the NLSW model by asymptotic analysis of the nondimensionalized integrated Navier–Stokes system with hydrostatic approximation is performed. We introduce the following averaged quantity for a generic function f depending on (t,x,y,z):

$$\bar{f}(t,x,y)=\frac{1}{h(t,x,y)}\int_d^{\xi} f(t,x,y,\eta)\,\mathrm{d}\eta.$$

From this, relation (4.1) can be directly expressed in the following form:

$$\partial_t h + \mathrm{div}_{xy}\big(h\overline{U}\big) = 0.$$

In the sequel, we assume the following asymptotic regime, as suggested in [1] and [12] from physical assumptions and asymptotic considerations (refer to [10] and [17]):

$$\frac{1}{R_e} = \varepsilon \nu_o, \quad k_l' = \varepsilon r_0 \quad \text{and} \quad k_t = \varepsilon r_1. \tag{5.1}$$

5.1. First order approximation

An asymptotic analysis of relations (3.5), (3.3) and (3.4) (see [1]) gives:

$$\partial_z^2 U = \mathrm{O}(\varepsilon), \quad \partial_z U_{|z=\xi} = \mathrm{O}(\varepsilon) \quad \text{and} \quad \partial_z U_{|z=d} = \mathrm{O}(\varepsilon). \tag{5.2}$$

This implies that:

$$U(t,x,y,z) = U(t,x,y,-d) + \mathrm{O}(\varepsilon),$$

and so

$$U(t,x,y,z) = \overline{U}(t,x,y) + \mathrm{O}(\varepsilon). \tag{5.3}$$

Furthermore, at first order, we immediately obtain the following equality:

$$\overline{U \otimes U}(t,x,y) = \overline{U} \otimes \overline{U}(t,x,y) + \mathrm{O}(\varepsilon). \tag{5.4}$$

The relations (5.1) enable us to write (3.11) under the form:

$$p(t,x,y,z) = p_o - \beta\kappa(t,x,y) + \frac{1}{F_r}(\xi - z) + \mathrm{O}(\varepsilon). \tag{5.5}$$

The mean curvature of the free surface, denoted by $\kappa(t,x,y)$, is usually expressed in term of the local surface elevation $\xi(t,x,y)$, in its non-dimensional form:

$$\kappa(\xi) = \frac{\varepsilon\xi_{xx}(1+\varepsilon^2\xi_y^2) - 2\varepsilon^3\xi_x\xi_y\xi_{xy} + \varepsilon\xi_{yy}(1+\varepsilon^2\xi_x^2)}{(1+\varepsilon^2\xi_x^2+\varepsilon^2\xi_y^2)^{3/2}}, \tag{5.6}$$

and then we have:

$$\kappa = \varepsilon\Delta\xi + \mathrm{O}\big(\varepsilon^3\big). \tag{5.7}$$

Thus, the pressure (5.5) reduces to:

$$p(t,x,y,z) = p_o + \frac{1}{F_r}(\xi - z) + \mathrm{O}(\varepsilon). \tag{5.8}$$

Using again the relations (5.1), and keeping in mind that $\partial_z\overline{U} = \mathrm{O}(\varepsilon)$, the relation (4.5) becomes at first order:

$$\partial_t\big(h\overline{U}\big) + \mathrm{div}_{xy}\big(h\overline{U}\otimes\overline{U}\big) + \nabla_{xy}\left(\int_d^\xi p\,\mathrm{d}z\right) + \frac{h\overline{U}^\perp}{R_o} = -r_0\overline{U}_{|z=d} - p_{|z=d}\nabla_{xy}d + p_o\nabla_{xy}\xi + \mathrm{O}(\varepsilon).$$

Plugging (5.5) into this expression yields therefore:

$$\nabla_{xy}\left(\int_d^\xi p\,\mathrm{d}z\right) = \frac{1}{2F_r}\nabla_{xy}\big(h^2\big) + p_o\nabla_{xy}h + \mathrm{O}(\varepsilon),$$

and

$$p_{|z=d}\nabla_{xy}d = \frac{1}{F_r}h\nabla_{xy}d + p_o\nabla_{xy}d + \mathrm{O}(\varepsilon).$$

Finally, dropping the O(ε) terms, we obtain the following NLSW model, which results of an approximation in O(ε) of the system (3.2)–(3.4), where we have set $\mathrm{div} = \mathrm{div}_{xy}$ and $\nabla = \nabla_{xy}$:

$$\begin{cases} \partial_t h + \mathrm{div}(h\overline{U}) = 0, & \text{(i)} \\ \partial_t(h\overline{U}) + \mathrm{div}(h\overline{U} \otimes \overline{U}) + \dfrac{1}{F_r^2} h\nabla h + (h\overline{U})^{\perp}/R_o = -r_0\overline{U} - \dfrac{1}{F_r^2} h\nabla d. & \text{(ii)} \end{cases} \tag{5.9}$$

Multiplying by HV^2/L and setting $\mathbf{u} = \overline{U}$, we recover the system in dimensionalized variables:

$$\begin{cases} \partial_t h + \mathrm{div}(h\mathbf{u}) = 0, & \text{(i)} \\ \partial_t(h\mathbf{u}) + \mathrm{div}(h\mathbf{u} \otimes \mathbf{u}) + gh\nabla h + f(h\mathbf{u})^{\perp} = -k_l\mathbf{u} - gh\nabla d. & \text{(ii)} \end{cases} \tag{5.10}$$

5.2. Second order approximation and parabolic correction

We perform here a parabolic correction in z for the horizontal velocity in order to take the vertical variability of the quantities into account and by this way improve the precision of the NLSW model (5.3). Coming back to the relation (3.5), we have for the horizontal velocity $U = (u, v)$:

$$\frac{\partial}{\partial z}\left(\frac{\nu_o}{\varepsilon}\frac{\partial U}{\partial z}\right) = \partial_t U + \mathrm{div}(U \otimes U) + \nabla p + \frac{U^{\perp}}{R_o} + \mathrm{O}(\varepsilon).$$

Considering the incompressibility relation, we have:

$$\frac{\partial}{\partial z}\left(\frac{\nu_o}{\varepsilon}\frac{\partial U}{\partial z}\right) = \partial_t U + U \cdot \nabla U + \nabla p + \frac{U^{\perp}}{R_o} + \mathrm{O}(\varepsilon), \tag{5.11}$$

and using the relations (5.3) and (5.5), (5.11) becomes:

$$\frac{\partial}{\partial z}\left(\frac{\nu_o}{\varepsilon}\frac{\partial U}{\partial z}\right) = \partial_t \overline{U} + \overline{U}\nabla\overline{U} + \frac{1}{F_r}\nabla\xi + \frac{\overline{U}^{\perp}}{R_o} + \mathrm{O}(\varepsilon),$$

so, keeping in mind that $\xi(t, x, y) = h(t, x, y) + d(x, y)$ and using Eq. (5.9 (ii)) we obtain:

$$\frac{\partial}{\partial z}\left(\frac{\nu_o}{\varepsilon}\frac{\partial U}{\partial z}\right) = -\frac{r_0}{h} U_{|z=d} + \mathrm{O}(\varepsilon).$$

Integrating from d to z we deduce:

$$\frac{\nu_o}{\varepsilon}\frac{\partial U}{\partial z} - \frac{\nu_o}{\varepsilon}\frac{\partial U}{\partial z}_{|z=d} = -\frac{r_0(z+d)}{h} U_{|z=d} + \mathrm{O}(\varepsilon).$$

Dropping the O(ε) terms, the boundary condition at the bottom (4.4) reads:

$$-\frac{\nu_o}{\varepsilon}\frac{\partial U}{\partial z}_{|z=d} = -r_0 U_{|z=-d} + \mathrm{O}(\varepsilon) \tag{5.12}$$

and we obtain:

$$\frac{\nu_o}{\varepsilon}\frac{\partial U}{\partial z} = r_0\left(1 - \frac{z+d}{h}\right) U_{|z=d} + \mathrm{O}(\varepsilon). \tag{5.13}$$

Then, integrating another time we deduce the expansion at second order in z of the horizontal velocity U:

$$U(t, x, y, z) = \left(1 + \frac{r_0\varepsilon(z+d)}{\nu_o}\left(1 - \frac{z+d}{2h}\right)\right) U_{|z=d} + \mathrm{O}(\varepsilon^2). \tag{5.14}$$

The expression of $\overline{U} = (\bar{u}, \bar{v})$ is then obtained, integrating (5.14) between $z = d$ and $z = \xi$:

$$\overline{U} = \left(1 + \frac{r_0\varepsilon h}{3\nu_o}\right) U_{|z=d} + \mathrm{O}(\varepsilon^2). \tag{5.15}$$

Furthermore, we have for the scalar horizontal component u:

$$u^2(t, x, y, z) = \left(1 + \frac{2\alpha_o\varepsilon(z+d)}{\nu_o}\left(1 - \frac{z+d}{2h}\right)\right) u^2_{|z=-d} + \mathrm{O}(\varepsilon^2),$$

and

$$\overline{u^2} = \frac{1}{h}\int_{-d}^{\xi} u^2 = \left(1 + \frac{2\alpha_o \varepsilon h}{3\nu_o}\right) u^2_{|z=-d} + \mathrm{O}(\varepsilon^2) = \bar{u}^2 + \mathrm{O}(\varepsilon^2).$$

It results that:

$$\overline{u^2} = \bar{u}^2 + \mathrm{O}(\varepsilon^2).$$

And in the same way we have the following tensorial equality:

$$\overline{U \otimes U} = \overline{U} \otimes \overline{U} + \mathrm{O}(\varepsilon^2). \tag{5.16}$$

Using (5.3), (5.7) and (5.1), the pressure (3.11) becomes:

$$p(t,x,y,z) = p_o + \frac{1}{F_r}(\xi - z) - \frac{2}{R_e}\partial_x \bar{u}(t,x,y) - \frac{2}{R_e}\partial_y \bar{v}(t,x,y) - \varepsilon\beta\Delta\xi + \mathrm{O}(\varepsilon^2). \tag{5.17}$$

The boundary condition at the bottom (4.4) is also simplified, recalling (5.1) and using (5.12):

$$\frac{1}{R_e}(\partial_z U)_{|z=d} = \mathrm{O}(\varepsilon^2).$$

Then, dropping the $\mathrm{O}(\varepsilon^2)$ terms we obtain:

$$\begin{aligned}&\left(-p\nabla_{xy}d + \frac{2}{R_e}D_h(U)\cdot\nabla_{xy}d - \frac{1}{R_e}\left(\frac{1}{\varepsilon^2}\partial_z U + \nabla_{xy}w\right)\right)_{|z=d}\\ &\quad = -r_0 U_{|z=d} - \varepsilon r_1 h|U|U_{|z=d} - \nabla_{xy}d\left(p - \frac{2}{R_e}\partial_z w\right)_{|z=d} + \mathrm{O}(\varepsilon^2),\end{aligned}$$

and so

$$\begin{aligned}&\left(-p\nabla_{xy}d + \frac{2}{R_e}D_h(U)\cdot\nabla_{xy}d - \frac{1}{R_e}\left(\frac{1}{\varepsilon^2}\partial_z U + \nabla_{xy}w\right)\right)_{|z=d}\\ &\quad = -r_0 U_{|z=-d} - \varepsilon r_1 h|U|U_{|z=d} - \nabla_{xy}d\left(p + \frac{2}{R_e}\operatorname{div}_{xy} U\right)_{|z=d} + g\mathrm{O}(\varepsilon^2),\end{aligned} \tag{5.18}$$

thanks to the incompressibility relation. Gathering (5.15), (5.17) and (5.18) and introducing:

$$\alpha_0(h) = \frac{r_0}{1 + \varepsilon r_0 h/(3\nu_o)}, \quad \alpha_1(h) = \frac{\varepsilon r_1}{(1 + \varepsilon r_0 h/(3\nu_o))^2} \quad \text{and} \quad \beta' = \varepsilon\beta$$

where $\alpha_0(h)$ and $\alpha_1(h)$ are modified friction coefficients, we obtain the following NLSW system, which results from an approximation in $\mathrm{O}(\varepsilon^2)$ in the asymptotic analysis of the system (3.2)–(3.4):

$$\begin{cases} \partial_t h + \operatorname{div}(h\overline{U}) = 0, \\ \partial_t(h\overline{U}) + \operatorname{div}(h\overline{U}\otimes\overline{U}) + \dfrac{1}{F_r^2}h\nabla h = -\alpha_0(h)\overline{U} - \alpha_1(h)h|\overline{U}|\overline{U} + \dfrac{1}{R_e}\operatorname{div}(h(\nabla\overline{U} + {}^t\nabla\overline{U})) \\ \quad + \dfrac{2}{R_e}\nabla(h\operatorname{div}\overline{U}) + \beta' h\nabla\Delta h - \dfrac{(h\overline{U})^\perp}{R_o} - \dfrac{1}{F_r^2}h\nabla d + \beta' h\nabla\Delta d. \end{cases} \tag{5.19}$$

Multiplying by HV^2/L and setting $\mathbf{u} = \overline{U}$, we recover the system in dimensionalized variables:

$$\begin{cases} \partial_t h + \operatorname{div}(h\mathbf{u}) = 0, \\ \partial_t(h\mathbf{u}) + \operatorname{div}(h\mathbf{u}\otimes\mathbf{u}) + gh\nabla h = -\alpha_0(h)\mathbf{u} - \alpha_1(h)h|\mathbf{u}|\mathbf{u} + 2\mu\operatorname{div}(hD(\mathbf{u})) + 2\mu\nabla(h\operatorname{div}\mathbf{u}) \\ \quad + \beta h\nabla\Delta h - f(h\mathbf{u})^\perp - gh\nabla d + \beta h\nabla\Delta d, \end{cases} \tag{5.20}$$

where we have:

$$\alpha_0(h) = \frac{k_l}{1 + k_l h/(3\mu)} \quad \text{and} \quad \alpha_1(h) = \frac{k_t}{(1 + k_l h/(3\mu))^2}.$$

6. Numerical experiments

In this section, we only present an academic test case in order to compare the results produced by the model developed here, with the particular viscous term $2\mu\,\mathrm{div}(hD(\mathbf{u})) + 2\mu\nabla(h\,\mathrm{div}\,\mathbf{u})$, with the results obtained with the viscous term $4\mu\,\mathrm{div}(hD(\mathbf{u}))$ extrapolated from the results obtained in the one-dimensional framework [1]. For this simulation, the Coriolis and the surface tension effects are neglected ($f = 0$ and $\beta = 0$). The numerical approximation is based on a hybrid Finite Volumes-Finite Differences method on a regular Cartesian mesh, introduced in [18]. The Finite Volume step aims at solving the hyperbolic system constituted of the convective terms and the bed-slope source term. It relies on a recent well-balanced method and is developed in [19]. The Finite Difference step aims at solving the system with the remaining source terms with a semi-implicit method, where the friction terms are discretized implicitly are the remaining terms explicitly [20].

6.1. The two-dimensional oblique dam-break problem

We study in this test the evolution of a mound of water over a flat bottom, which is suddenly released from an initial position, generating a bore-like wave such that the front of the water propagates with an inclination of 45° with respect to the boundaries of the computational domain. The base of this domain is a $[-0.5, 0.5] \times [-0.5, 0.5]$ square. The initial condition is defined as follows:

$$h(0, x, y) = \begin{cases} h_L^0 & \text{for } x + y \leqslant 0, \\ h_L^0/2 & \text{otherwise} \end{cases} \tag{6.1}$$

and

$$\mathbf{u}(0, x, y) = 0, \tag{6.2}$$

and is plotted on Fig. 3. Flow properties computed on the central cross-section orthogonal to the propagating front (the $x = y$ plane) with the new viscous term $2\mu\,\mathrm{div}(hD(\mathbf{u})) + 2\mu\nabla(h\,\mathrm{div}\,\mathbf{u})$ are compared to the results obtained with the classical viscous term $4\mu\,\mathrm{div}(hD(\mathbf{u}))$, assuming that the effects induced by the boundaries can be neglected for this section and for a small time of evolution. This assumption is clearly not true for the sections close to the boundary. Values of $\Delta x = \Delta y = 0.01$ and $CFL = 0.7$ have been used for this test and the initial condition h_L^0 is set to 1. We have also neglected the friction effects ($k_l = k_t = 0$) and the viscosity μ is set to 0.1.

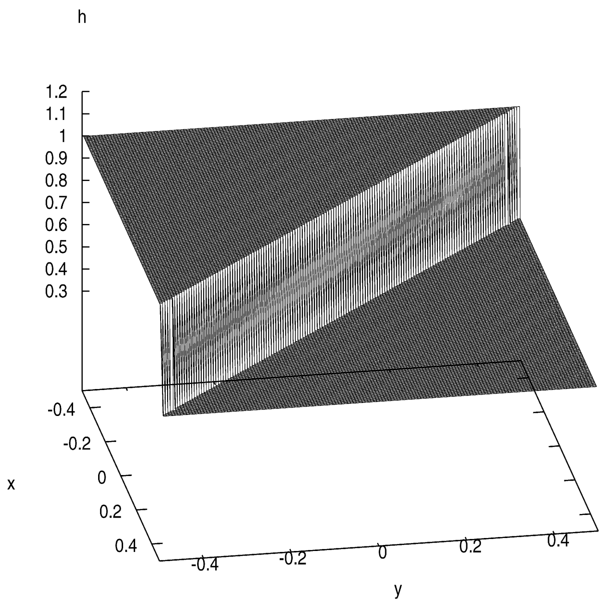

Fig. 3. The two-dimensional oblique dam-break problem on a flat bottom. Initial condition for $h_L^0 = 1$ and $h_R^0 = h_L^0/2$.

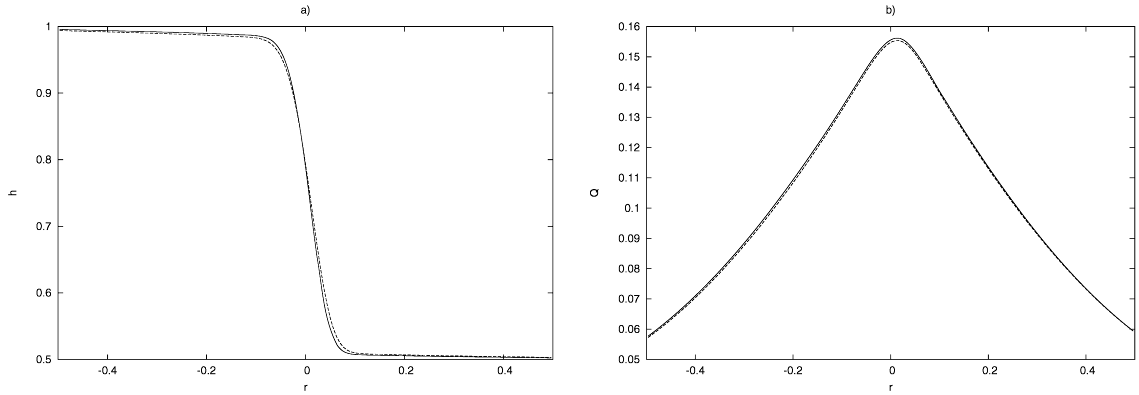

Fig. 4. The two-dimensional oblique dam-break problem on a flat bottom for $h_L^0 = 1$ and $h_R^0 = h_L^0/2$. Comparison between numerical results obtained with the new viscous term (solid lines) and the expected term from the one-dimensional case (dashed lines) for: (a) the water depth h and (b) the discharge $Q = (h\mathbf{u})$ at $t = 0.03$ s.

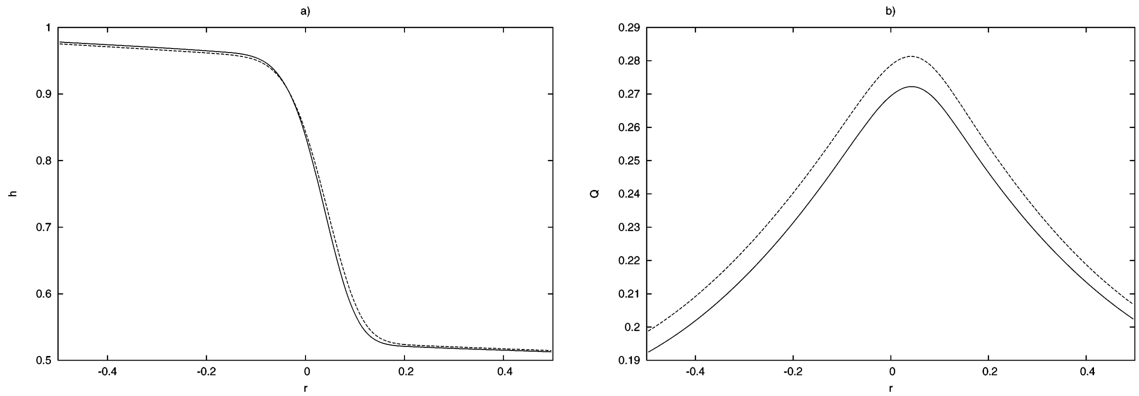

Fig. 5. The two-dimensional oblique dam-break problem on a flat bottom for $h_L^0 = 1$ and $h_R^0 = h_L^0/2$. Comparison between numerical results obtained with the new viscous term (solid lines) and the expected term from the one-dimensional case (dashed lines) for: (a) the water depth h versus the radius r and (b) the discharge $Q = (h\mathbf{u})$ versus the radius r, at $t = 0.09$ s.

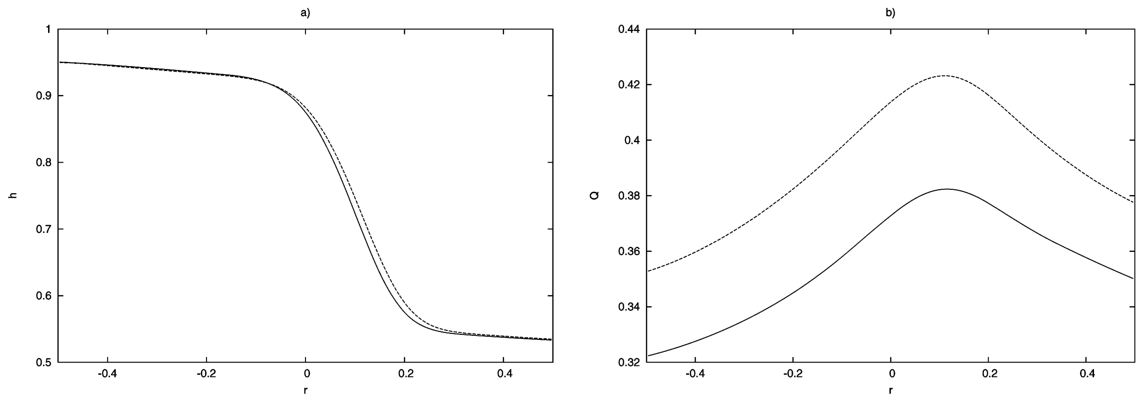

Fig. 6. The two-dimensional oblique dam-break problem on a flat bottom for $h_L^0 = 1$ and $h_R^0 = h_L^0/2$. Comparison between numerical results obtained with the new viscous term (solid lines) and the expected term from the one-dimensional case (dashed lines) for: (a) the water depth h and (b) the discharge $Q = (h\mathbf{u})$ at $t = 0.18$ s.

This comparison is reported on Figs. 4–6 at three times during the evolution before the boundaries effects become too important. We can observe on Fig. 4(a) that the front of the two numerical solutions remain close at the beginning of the propagation and increases with respect to time (see Fig. 6(a)). The differences in viscous effects are more rapidly observable on the discharge profiles, leading to a shift of magnitude which becomes greater with respect to time. It is worth mentioning that further numerical simulations have highlighted that this behavior is confirmed and amplified as μ is increasing. However, these are only preliminary computing and more realistic simulations are obviously needed. In particular, a comparison with some experiments concerning the dam-break problem over a dry bed (refer to Dressler [21]) is actually under study.

7. Conclusion

Improving the accuracy of the analysis yields a new NLSW model. The new viscosity formulation derived here is not $4\mu\,\mathrm{div}(hD(\mathbf{u}))$ as it may be extrapolated from the results obtained in [1] in the one-dimensional case but the new term $2\mu\,\mathrm{div}(hD(\mathbf{u})) + 2\mu\nabla(h\,\mathrm{div}\,\mathbf{u})$. It differs from [9] by the addition of the symmetric term $2\mu\nabla(h\,\mathrm{div}\,\mathbf{u})$. It is worth mentioning that this new viscous formulation does not belong to the frame with extra mathematical entropy recently introduced by Bresch and Desjardins in [22]. However, this new two-dimensional model is consistent with the one-dimensional system previously introduced in [1] since Eqs. (5.19) degenerate into the system obtained in [1]. The addition of a quadratic drag term with modified water depth dependent coefficients is coherent with the usual friction formulation used in oceanographic simulations, namely the Manning–Chezy formulation.

In the recent article [12] Ferrari and Saleri also uses the main ideas of [1] to derive a two-dimensional NLSW model. However the authors added in [12] a standard viscous term after deriving the model, and the term does not match the viscous term of [1] in the degenerated one-dimensional case. We emphasize that in the derivation proposed here, the new viscous term is asymptotically derived from the Navier–Stokes–Coriolis equations and degenerates into the same term as in [1]. Using this new viscous formulation and a new well-balanced finite volume discretization of this NLSW model, two-dimensional numerical simulations of topographically controlled breaking wave induced large scale eddies and vortices evolution in the surf and swash zone, inspired from [23], are currently under study.

Furthermore, we also justify here the introduction of a small third-order capillary term $h\nabla\Delta h$. As it has been emphasized in [9] and [8], this term is of a great utility to prove the global existence of weak solutions for this new NLSW model, without additional assumptions on the positivity of the water depth, since it provides a useful H^2 estimation. This problematic is addressed in [18].

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